Economics Letters 117 (2012) 525–527

Contents lists available at SciVerse ScienceDirect

Economics Letters

journal homepage: www.elsevier.com/locate/ecolet

Optimal beliefs in the long run: An overlapping generations perspective

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ARTICLE INFO

Article history:
Received 9 April 2012
Accepted 29 June 2012
Available online 6 July 2012

JEL classification:
D03
D83
D84
D91
G11

Keywords:
Optimal expectations
Overlapping generations
Behavioral decision theory

ABSTRACT

People have the natural tendency to be optimistic and believe that good outcomes in the future are more likely, but also want to avoid overestimation that could result in bad decision-making. Brunnermeier, Brunnermeier and Parker (2005) and Brunnermeier et al. (2007) established an optimal beliefs framework that balances these two incentives. This paper follows and extends the optimal beliefs framework to consider optimal beliefs in the long run in an overlapping generations sense. Assuming no short-selling, results show that, in almost all cases, there does not exist a stable and interior long-term optimal belief.

1. Introduction

In standard utility theory, utility functions generally contain states and actions as arguments. Beliefs do not affect people's sense of well-being, but rather only enter an agent's decision making process via maximizing the expected utility function. In fact, since Muth (1961) almost all research has adopted the rational expectations assumption that subjective beliefs and objective probabilities coincide.

However, a substantial body of psychological research shows that beliefs do affect people's sense of well-being in a very direct way. Brunnermeier and Parker (2005) assume that forward-looking agents care about expected future utility flow, and hence have higher instantaneous well-being if they are optimistic about the future. The optimal expectations framework established in Brunnermeier and Parker (2005) involves a two-stage decision making process. In stage 1, agent chooses "optimally" subjective beliefs subject to the *optimal* actions of stage 2. In stage 2, the agent solves the portfolio allocation problem given *subjective* beliefs. Beliefs impact instantaneous well-being directly through anticipatory emotions of the future flow utility and indirectly through their effect on portfolio allocations. This result is consistent with the abundance of psychological research.

Our model of beliefs follows and extends the optimal expectations framework of Brunnermeier and Parker (2005). We believe that if the first generation of agents, G_1, derive subjective beliefs to maximize their well-being—namely the average of their expected present discounted value of utility flows, subsequent generations will do so as well in every period over time. The natural question is what would be the agents' subjective belief in the long run? We assume that in period T, generation G_T cohorts derive their Tth period subjective beliefs based on their Tth period objective probabilities. And in period $T+1$, generation G_T's children take their parents' Tth period subjective beliefs as their $(T+1)$th period objective probabilities and derive their own $(T+1)$th period subjective beliefs.

2. Model

We consider a world where the uncertainty can be described by 2 states where no short-selling is allowed. In an overlapping generations framework, generation G_T of cohorts choose optimal beliefs so as to maximize their well-being function:

$$\mathcal{W} = \sum_{s=1}^{2} \hat{\pi}_{s,T}(1 - e^{-\alpha c^{\star}_{s,T}(\hat{\boldsymbol{\pi}}_{\boldsymbol{T}})}) + \sum_{s=1}^{2} \pi_{s,T}(1 - e^{-\alpha c^{\star}_{s,T}(\hat{\boldsymbol{\pi}}_{\boldsymbol{T}})}), \quad (1)$$

where

(a) $c^{\star}_{s,T}(\hat{\boldsymbol{\pi}}_{\boldsymbol{T}})$ is obtained through maximizing the expected utility function given subjective beliefs:

$$\max_{(c_{1,T}, c_{2,T})} \; [\hat{\pi}_{1,T} u(c_{1,T}) + \hat{\pi}_{2,T} u(c_{2,T})] \quad \text{subject to}$$
$$p_1 c_{1,T} + p_2 c_{2,T} = 1 \quad \text{and} \quad c_{1,T}, c_{2,T} \geq 0, \quad (2)$$

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doi:10.1016/j.econlet.2012.06.052

(b) objective probabilities are inherited from the previous generation:

$$\pi_{1,T} = \hat{\pi}_{1,T-1} \quad \text{and} \quad \pi_{2,T} = \hat{\pi}_{2,T-1}, \quad \forall T = 2, 3, \ldots. \tag{3}$$

Details are as follows:

Overlapping generations of agents. Suppose that $t = 1, 2, \ldots,$ and that at every time t there is born a new generation G_t of individuals who live for two periods. There does not exist an initial generation G_0 at around $t = 0$ that lives for only 1 period. Every generation consists of a number of homogeneous agents, and assume that population growth rate is zero. Therefore, the number of agents in every generation can be normalized to 1. Agents have two-period economic lives and do not care about future generations. Hence there are no bequests, no dynastic behavior, no altruism.

At some time period T, consider the generation G_T. In the first period ($t = T$), agents form optimal beliefs $(\hat{\pi}_{1,T}, \hat{\pi}_{2,T})$ and allocate their portfolios $(c^*_{1,T}, c^*_{2,T})$, based on the objective probabilities they perceive of the 2 states of the world $(\pi_{1,T}, \pi_{2,T})$. In the second period ($t = T+1$), agents receive the realized profits of their portfolio allocation choices and die by the end of the second period.

For every generation, agents pass their optimal beliefs on to their descendants, and descendants regard their parents' optimal beliefs as objective probabilities of the world. Mathematically, $\pi_{1,T} = \hat{\pi}_{1,T-1}$ and $\pi_{2,T} = \hat{\pi}_{2,T-1}$, $\forall T = 2, 3, \ldots$. In the initial period $t = 1$, the first generation G_1 will be given objective probabilities $\pi_{1,0}$ and $\pi_{2,0}$.

Utility function. Assume an investor has the exponential utility function $u(c) = 1 - e^{-\alpha c}$, where $\alpha > 0$. The exponential utility function exhibits absolute risk-aversion behavior, so our assumption is that the agents' attitude towards risk is invariant with the amount of wealth they have accumulated. Therefore, in the agents' budget constraint, $c_1 p_1 + c_2 p_2 = w$, we can conveniently normalize their wealth to 1 and re-write the budget constraint as $c_1 p_1 + c_2 p_2 = 1$, without loss of generality.

Well-being function. Each agent's beliefs maximize his well-being, defined as the average expected utility across period T and period $T + 1$ when actions are optimal given subjective beliefs. That is, $\hat{\pi}$ maximizes $\frac{1}{2}E[V_T + V_{T+1}]$ subject to the constraints that the $\hat{\pi}_{1,T}$ and $\hat{\pi}_{2,T}$ are probabilities and that portfolio choices are optimal given $\hat{\pi}_{1,T}$ and $\hat{\pi}_{2,T}$. Because the economy has 2 states, in order to determine the optimal beliefs, the investor only needs to maximize $\mathcal{W} = \sum_{s=1}^{2} \hat{\pi}_{s,T} u(c^\star_{s,T}) + \sum_{s=1}^{2} \pi_{s,T} u(c^\star_{s,T})$ with respect to $\hat{\pi}_{1,T}$ and $\hat{\pi}_{2,T}$.

With such a well-being function, beliefs impact well-being directly through anticipation of future utility and indirectly through their effect on portfolio choices.

Portfolio choices. An agent's optimal portfolio allocation choices, $(c^\star_{1,T}, c^\star_{2,T})$, maximize his expected utility given his subjective beliefs, $(\hat{\pi}_{1,T}, \hat{\pi}_{2,T})$.

Formally, $(c^\star_{1,T}, c^\star_{2,T})$ are obtained though the following:

$$\max_{(c_{1,T}, c_{2,T})} [\hat{\pi}_{1,T} u(c_{1,T}) + \hat{\pi}_{2,T} u(c_{2,T})] \quad \text{subject to}$$

$$p_1 c_{1,T} + p_2 c_{2,T} = 1 \quad \text{and} \quad c_{1,T}, c_{2,T} \geq 0, \tag{4}$$

where p_1, $p_2 > 0$ are the prices of the Arrow–Debreu security yielding one unit in state 1 and 2, respectively.

3. Results and discussions

3.1. Optimal portfolio choices

Proposition 1 (*Existence and Uniqueness of Optimal Portfolio Choices*). *Assuming no short-selling, at any period t, the generation G_t cohorts' optimal portfolio choices, c^*_{1t} and c^*_{2t}, exist and are unique.*

$$(c^*_{1t}, c^*_{2t}) = \begin{cases} \left(0, \dfrac{1}{p_2}\right) & \text{if } \hat{\pi}_{1t} \leqslant \dfrac{p_1}{p_1 + p_2 e^{\alpha/p_2}} \\ \left(\dfrac{p_2 \log\left(\frac{\hat{\pi}_{1t}}{1-\hat{\pi}_{1t}} \frac{p_2}{p_1}\right) + \alpha}{\alpha(p_1+p_2)}, \dfrac{p_1 \log\left(\frac{1-\hat{\pi}_{1t}}{\hat{\pi}_{1t}} \frac{p_1}{p_2}\right) + \alpha}{\alpha(p_1+p_2)}\right) & \\ \qquad \text{if } \dfrac{p_1}{p_1 + p_2 e^{\alpha/p_2}} < \hat{\pi}_{1t} < \dfrac{p_1}{p_1 + p_2 e^{-\alpha/p_1}} & \\ \left(\dfrac{1}{p_1}, 0\right) & \text{if } \hat{\pi}_{1t} \geqslant \dfrac{p_1}{p_1 + p_2 e^{-\alpha/p_1}}. \end{cases}$$

3.2. Optimal subjective beliefs

Proposition 2 (*Existence of Optimal Beliefs*). *At any period t, the generation G_t cohorts' optimal portfolio beliefs, $\hat{\pi}^*_{1t}$ and $\hat{\pi}^*_{2t}$, exist and, in almost all cases, are unique.*

In order to break down the proof of this proposition, we need several lemmas. Notice that because optimal portfolio choices c^*_{1t} and c^*_{2t} are piecewise functions with three pieces, the well-being function will need to be expressed as a piecewise function as well.

Lemma 1. *The well-being function can be expressed as a piecewise function with three pieces.*

Proof.

$$\mathcal{W} = \begin{cases} (\hat{\pi}_1 + \pi_1)(1 - e^{-\alpha/p_1}), \ \hat{\pi}_{1t} \in \left[\dfrac{p_1}{p_1 + p_2 e^{-\alpha/p_1}}, 1\right] \\ (\hat{\pi}_1 + \pi_1)\left[1 - e^B \left(\dfrac{\hat{\pi}_1}{1 - \hat{\pi}_1}\right)^A\right] \\ \qquad + (2 - \hat{\pi}_1 - \pi_1)\left[1 - e^D \left(\dfrac{\hat{\pi}_1}{1 - \hat{\pi}_1}\right)^C\right], \\ \qquad \hat{\pi}_{1t} \in \left(\dfrac{p_1}{p_1 + p_2 e^{\alpha/p_2}}, \dfrac{p_1}{p_1 + p_2 e^{-\alpha/p_1}}\right) \\ (2 - \hat{\pi}_1 - \pi_1)(1 - e^{-\alpha/p_2}), \ \hat{\pi}_{1t} \in \left[0, \dfrac{p_1}{p_1 + p_2 e^{\alpha/p_2}}\right]. \end{cases}$$

where $A = \frac{-p_2}{p_1+p_2}$, $B = \frac{-p_2 \log(\frac{p_2}{p_1}) - \alpha}{p_1+p_2}$, $C = \frac{p_1}{p_1+p_2}$, and $D = \frac{p_1 \log(\frac{p_2}{p_1}) - \alpha}{p_1+p_2}$. □

Given the well-being function being piecewise, we find the global optimal belief in three steps. Notice that the well-being function has three pieces. The subscripts of $\mathcal{W}^*$ indicates the maximum well-being in a particular subinterval. We label the optimal beliefs in every piece as $\mathcal{W}^*_1$, $\mathcal{W}^*_2$, and $\mathcal{W}^*_3$, corresponding to the subinterval $\hat{\pi}_{1t} \in \left[0, \frac{p_1}{p_1+p_2 e^{-\alpha/p_1}}\right]$, $\left(\frac{p_1}{p_1+p_2 e^{\alpha/p_2}}, \frac{p_1}{p_1+p_2 e^{-\alpha/p_1}}\right)$, and $[\frac{p_1}{p_1+p_2 e^{\alpha/p_2}}, 1]$, respectively.

Lemma 2. $\mathcal{W}^*_1 = (2 - \pi_1)(1 - e^{-\alpha/p_2})$, *and* $\mathcal{W}^*_3 = (1 + \pi_1)(1 - e^{-\alpha/p_1})$.

To find $\mathcal{W}_2$, we need two additional lemmas.

Lemma 3. *The first derivative of $\mathcal{W}^*_2$ is* $\frac{\partial \mathcal{W}}{\partial \hat{\pi}_1} = \left[\frac{(\frac{\hat{\pi}_1}{1-\hat{\pi}_1})^A e^B}{\hat{\pi}_1 (1-\hat{\pi}_1)^2 (1+p)}\right] \cdot$ $[-(1+p)^2 \hat{\pi}_1^3 + (1+p)(2+p)\hat{\pi}_1^2 - (1+2p)\hat{\pi}_1 + p\pi_1]$, *where $p = p_2/p_1$ denotes the price ratio.*

Lemma 4. *The second derivative of* $\mathcal{W}_2$ *is* $\frac{\partial^2 \mathcal{W}}{\partial \hat{\pi}_1^2} = \frac{(\frac{\hat{\pi}_1}{1-\hat{\pi}_1})^A e^B p}{(1-\hat{\pi}_1)^3 \hat{\pi}_1^2 (1+p)^2}$ $[-3(1+p)\hat{\pi}_1^2 + (1+2p+3(1+p)\pi_1)\hat{\pi}_1 - (1+2p)\hat{\pi}_1]$.

Put the above two lemmas together, and we can find $\mathcal{W}_2^*$:

Lemma 5. $\mathcal{W}_2^*$ *exists and can be solved analytically.*

The existence of $\mathcal{W}_2^*$ is established by the the function $\mathcal{W}_2$'s continuity and the compactness of its interval. To solve for $\mathcal{W}_2^*$ analytically, we need to find the roots of the first derivative of $\mathcal{W}_2$. Then we apply the second derivative test to find those local maxima in the range $\hat{\pi}_{1t} \in \left(\frac{p_1}{p_1+p_2 e^{\alpha/p_2}}, \frac{p_1}{p_1+p_2 e^{-\alpha/p_1}}\right)$. For those local maxima in the range, we find the one that gives the largest $\mathcal{W}_2$ and denotes the well-being associated with this subjective belief as $\mathcal{W}_2^*$.

After obtaining $\mathcal{W}_1^*$, $\mathcal{W}_2^*$, and $\mathcal{W}_3^*$ by using Lemmas 2 and 5, we can compare the three maximum well-beings in each segment and find the global maximum well-being. The optimal subjective belief is the distorted belief associated with the global maximum well-being.

Corollary 1. *Given fixed* p_1, p_2 *and* π_1, *as we increase* α, *the global optimal belief is more likely to be interior. In other words, there exists* α_0 *such that the subjective belief associated with* $\mathcal{W}_2^*$ *is the optimal belief the agents adopt, for all* $\alpha > \alpha_0$.

3.3. Long run optimal subjective beliefs

After the agents in generation G_T set their optimal subjective beliefs in period T, their descendants, generation G_{T+1}, are born at the beginning of period $T+1$. For agents in generation G_{T+1}, they take the optimal subjective beliefs set by generation G_T in period T as their objective probabilities of the states of world. They will then form their own distorted beliefs, maximize their well-being function, and decide on their own optimal subjective beliefs. This iteration keeps on and there will be some type of the long-term optimal subjective beliefs.

Proposition 3. *There does not exist a stable and interior long term optimal belief,* $\hat{\pi}^*$.

Proof. Suppose we have an interior long run optimal subjective belief, say $\hat{\pi}^*$. Then, because descendants inherit objective probabilities from their parents, we know that $\hat{\pi}^* = \hat{\pi}_1 = \pi_1$ in the long run. Apply this to Lemma 3, which can be rewritten as

$$-(1+p)^2(\hat{\pi}^*)^3 + (1+p)(2+p)(\hat{\pi}^*)^2$$
$$-(1+2p)\hat{\pi}^* + p\hat{\pi}^*. \quad (5)$$

The roots to this degree 3 polynomial are $\hat{\pi}^* = 0$, $\hat{\pi}^* = \frac{1}{1+p}$, and $\hat{\pi}^* = 1$. Since $\hat{\pi}^*$ is interior, the only feasible candidate for stable and interior long run optimal belief is $\hat{\pi}^* = \frac{1}{1+p}$. □

Use the second derivative to check if $\hat{\pi}^* = \frac{1}{1+p}$ is indeed a maximum. We check the sign of the second derivative as in Lemma 4. The roots are $\hat{\pi}_1 = \pi_1$ and $\hat{\pi}_1 = \frac{1+2p}{3(1+p)}$. There are two cases:

(1) If $p = 1$, then $\frac{1}{1+p} = \frac{1+2p}{3(1+p)}$, and the second derivative is always non-positive. Hence $\hat{\pi}^* = \frac{1}{1+p}$ is indeed a local maximum.

(2) If $p \neq 1$, then $\frac{1}{1+p} \neq \frac{1+2p}{3(1+p)}$. Since the second derivative at $\hat{\pi}^* = \frac{1}{1+p}$ is 0 and the third derivative at $\hat{\pi}^* = \frac{1}{1+p}$ is nonzero, we know that $\hat{\pi}^* = \frac{1}{1+p}$ is an inflection point and cannot be a local maximum.

Hence, unless $p = 1$ and $\hat{\pi}^* = \frac{1}{2}$, in all other cases there does not exist a stable and interior long term optimal belief.

4. Conclusion

This paper follows the optimal expectations framework established by Brunnermeier and Parker (2005) and Brunnermeier et al. (2007) but takes a long-term perspective when investigating optimal beliefs. Assuming that no short-selling is allowed, results show that: (1), at any particular time period t, the generation G_t cohorts always have unique optimal portfolio choices as well unique optimal beliefs; and (2), in the long run, except for a special case in which the two states of the world are identical and that agents start with unbiased objective beliefs about the two states, in all other cases there does not exist a stable and interior long run optimal belief. Depending on the price ratio, the agents in sufficiently late generations will always make extreme choices of betting all their wealth into one state. Further research can be done to see if these results are robust to different utility functions. We leave the robustness test for the causes as future research.

Acknowledgments

I am grateful to Paulo Natenzon, Markus Brunnermeier, Costas Azariadis, Sebastian Galiani, and Bruce Petersen for invaluable comments.

References

Brunnermeier, M.K., et al., 2007. Optimal beliefs, asset prices, and the preference for skewed returns. AER 97, 159–165.
Brunnermeier, M.K., Parker, J.A., 2005. Optimal expectations. AER 95, 1092–1118.
Muth, J.F., 1961. Rational expectations and the theory of price movements. Econometrica 29, 315–335.

Further reading

Caplin, A., Leahy, J., 2001. Psychological expected utility theory and anticipatory feelings. QJE 116, 55–79.
Loewenstein, G., 1987. Anticipation and the valuation of delayed consumption. Economic Journal 97, 666–684.